Distributed Consensus on Manifolds using the Riemannian Center of Mass

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motivation for consensus on manifolds

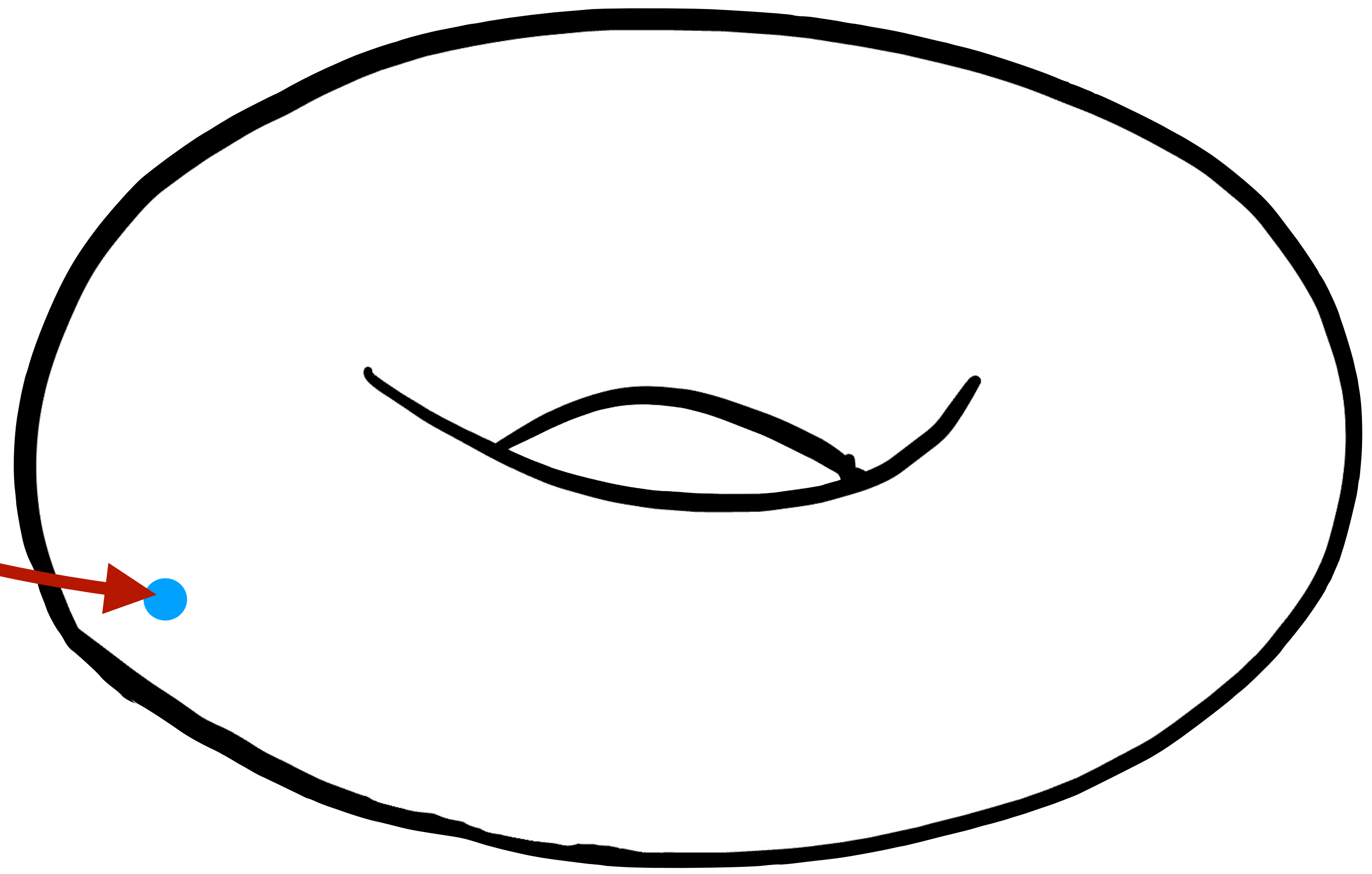
- consensus is central to **distributed computation**
  - Steer a set of agents to a **single point**
  - Studied mostly on Euclidean spaces

The Canadarm2 has a **non-Euclidean** state space.

- Robots with **non-Euclidean** state spaces

- **Formation control** on manifolds

The state space of a dome camera is a **torus** $\mathbb{S}^1 \times \mathbb{S}^1$.

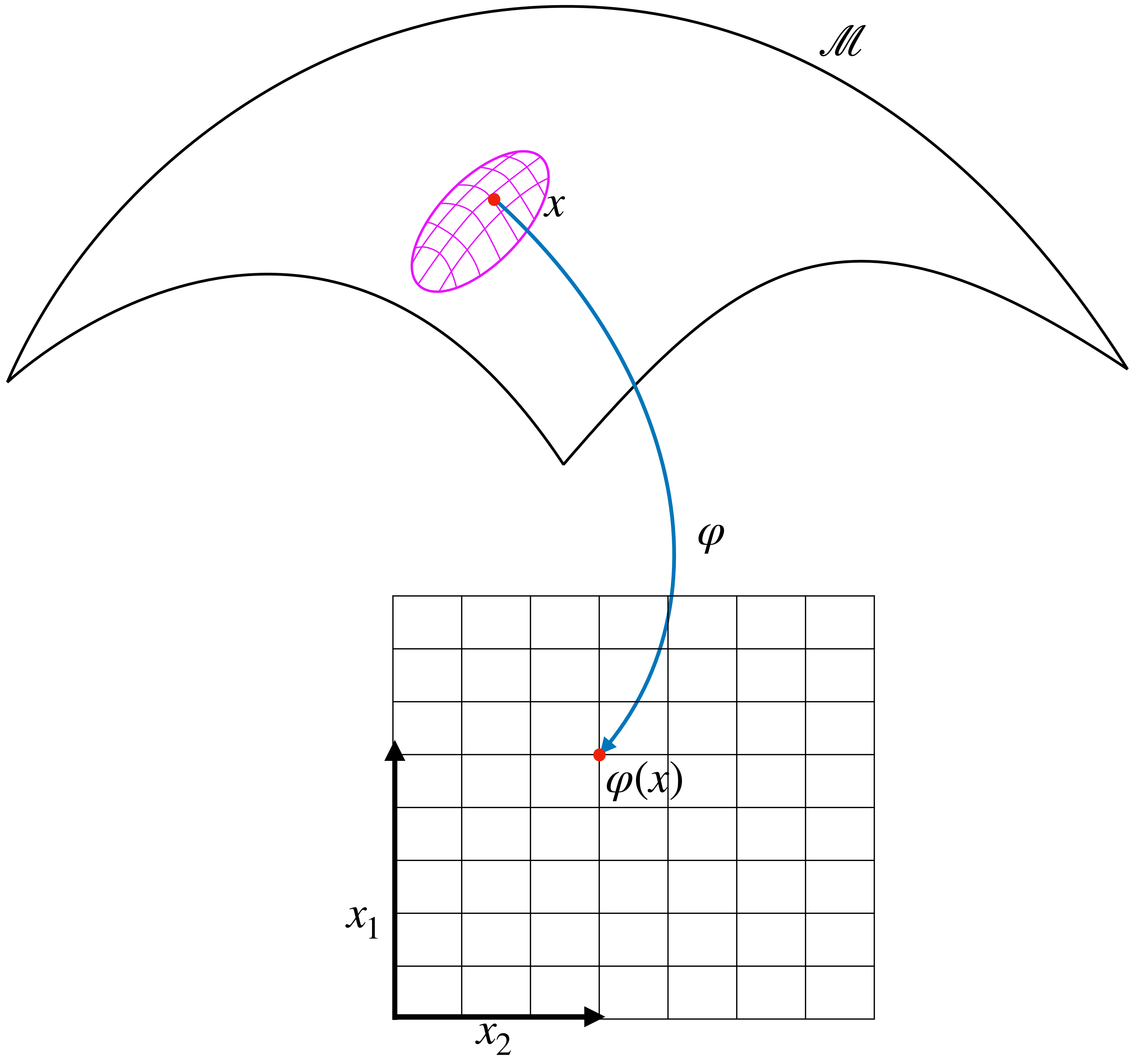


background

- **smooth manifold:**
  - topological space $\mathscr{M} \subset \mathbb{R}^n$
  - locally Euclidean

- **Riemannian metric:**
  - induces metric space structure $d_g(\,.\,,\,.\,)$

- intrinsic vs extrinsic quantities



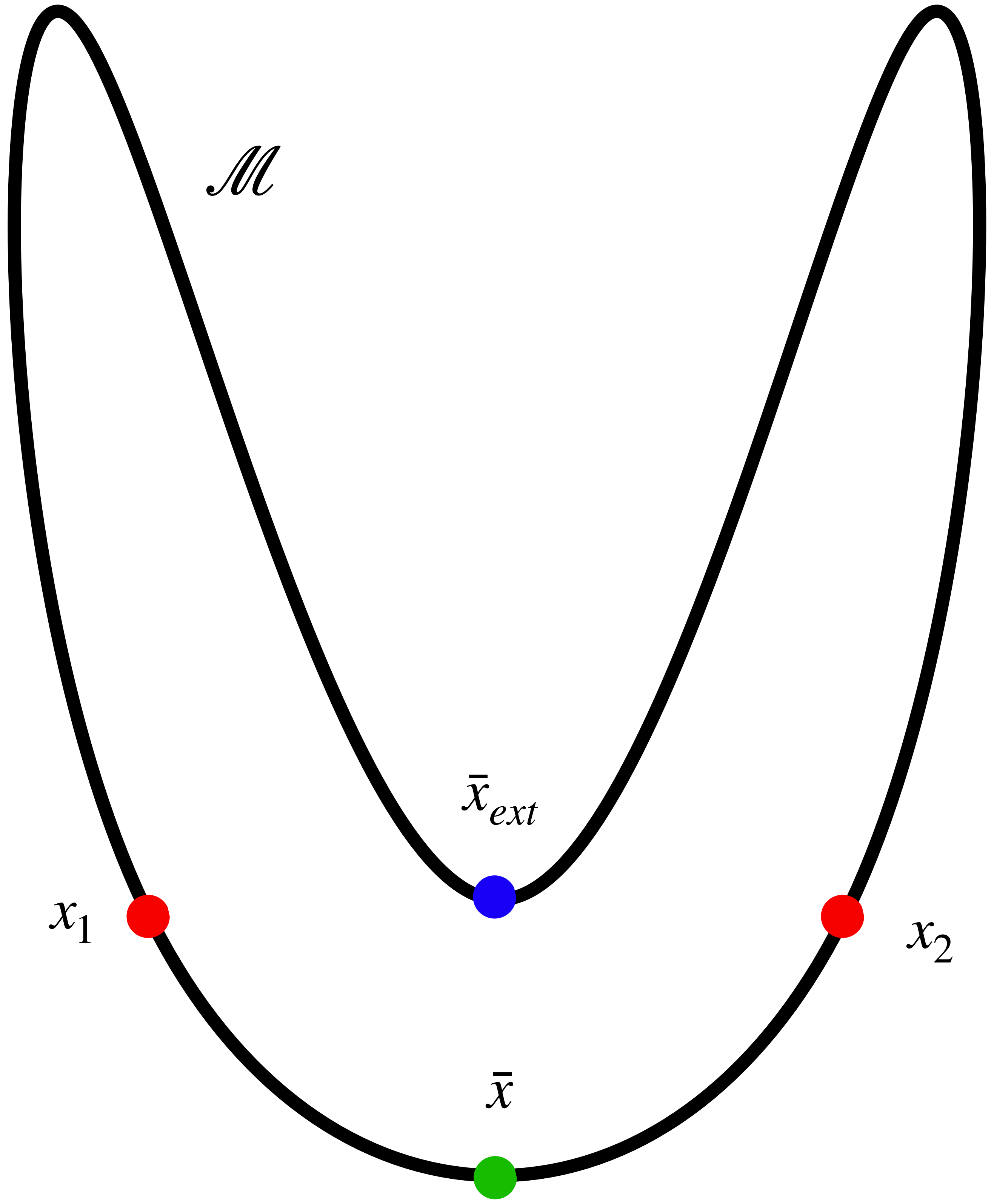
background

what is an average on a Riemannian manifold?

- average solves $\min_{x \in \mathbb{R}^n} \sum_{i=1}^{N} \|x - z_i\|^2$
- Riemannian center of mass (RCM) $\bar{z}$ solves

$$\min_{x \in \mathscr{M}} f(x) := \sum_{i=1}^{n} d_g(x, z_i)^2$$

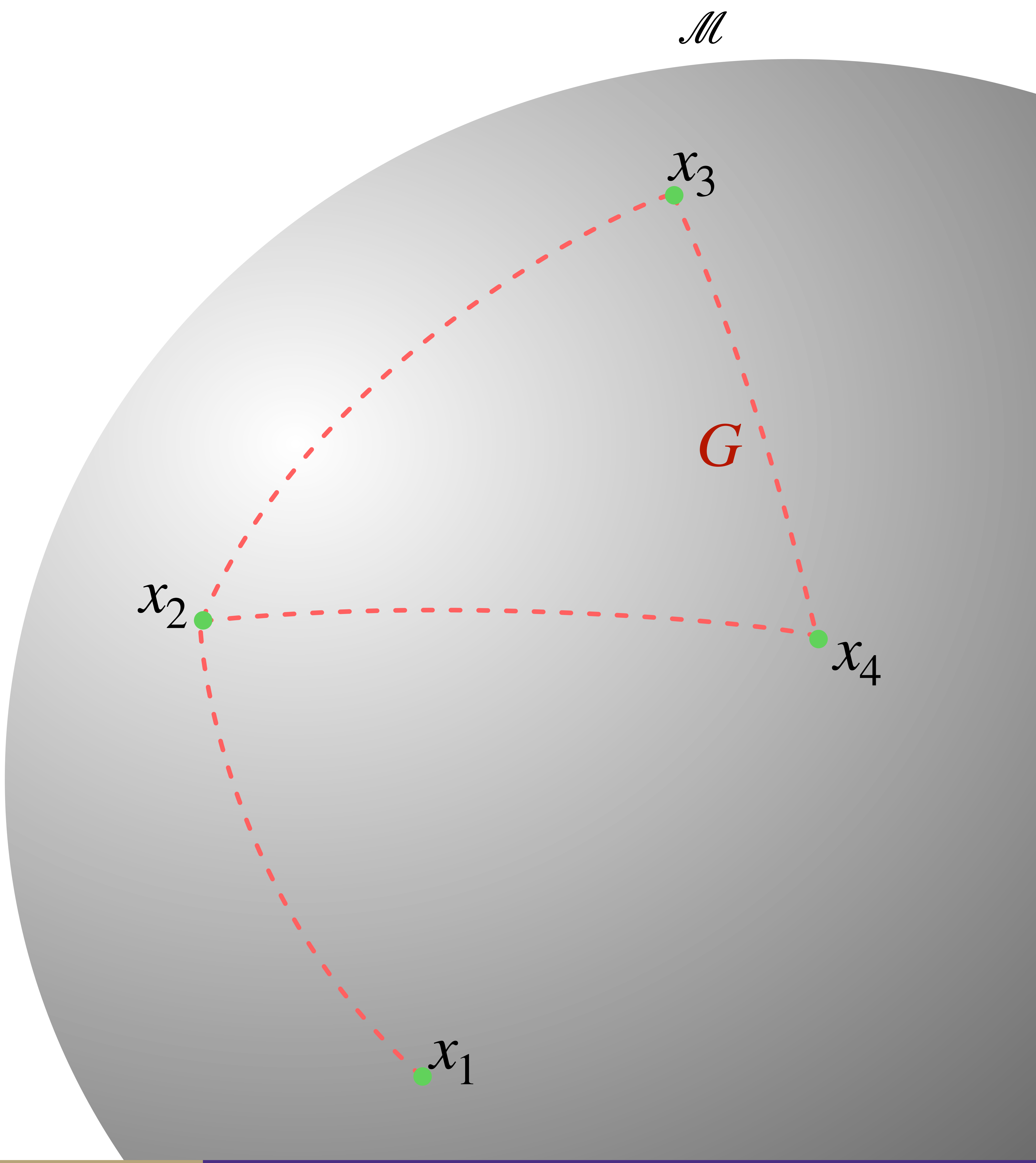
- intrinsic better than extrinsic: $\bar{z}_{ext} = \text{proj}_{\mathscr{M}}\left(\frac{1}{N}\sum_{i=1}^{N} z_i\right)$
- applications
  - medical imaging
  - weather



The RCM (green) vs. extrinsic mean (blue) of a pair of points (red) [Afsari, 2011]

consider:

- Riemannian manifold $\mathscr{M}$
- agents $x_1, \ldots, x_N \in \mathscr{M}$
  - communication graph G
  - $x_i(k+1) = F_i(k, x_m(k) : m \sim i)$

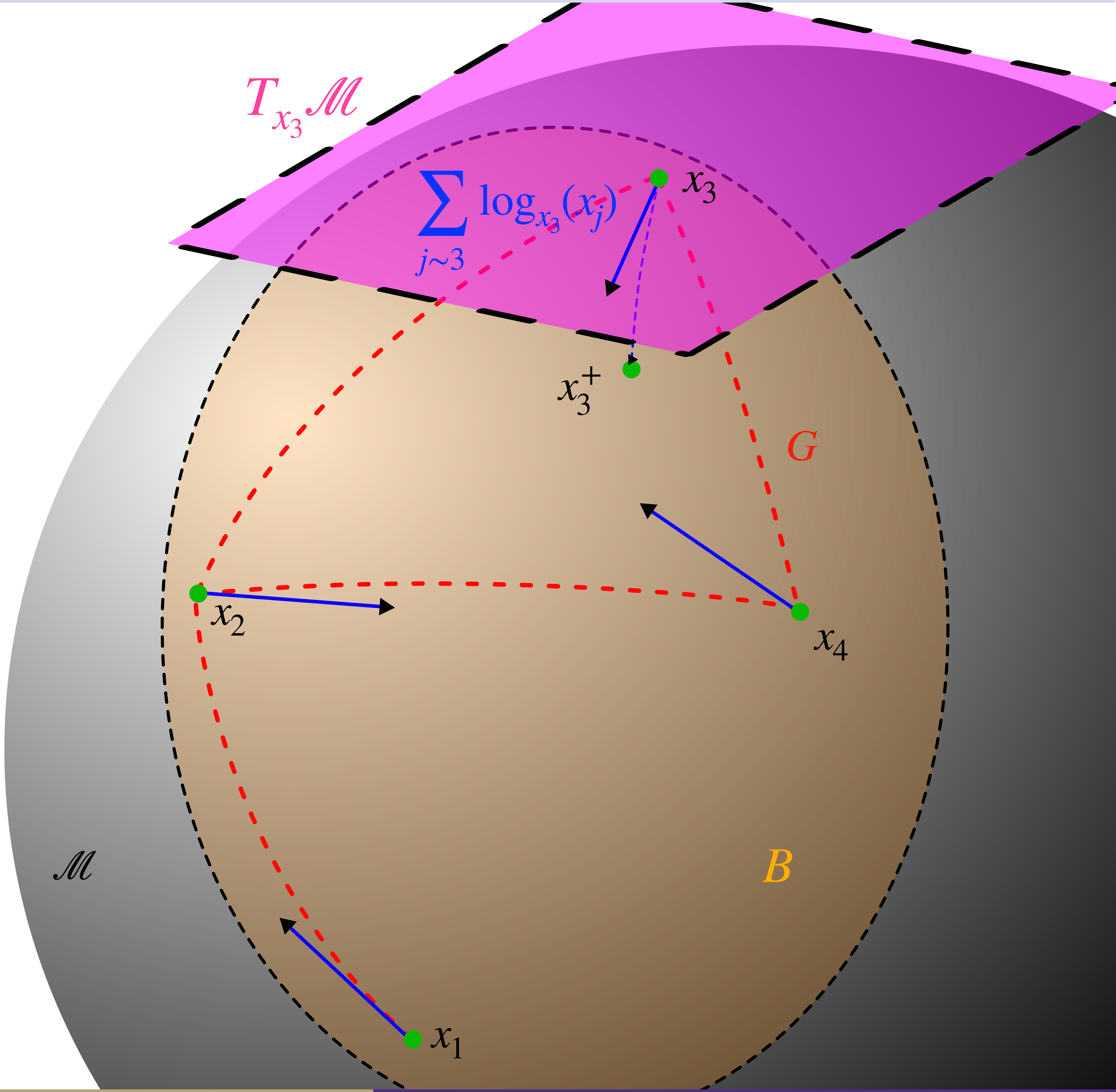


problem:

- design **intrinsic** + **distributed** dynamics to **synchronize** agents
  - no projection, no embedding

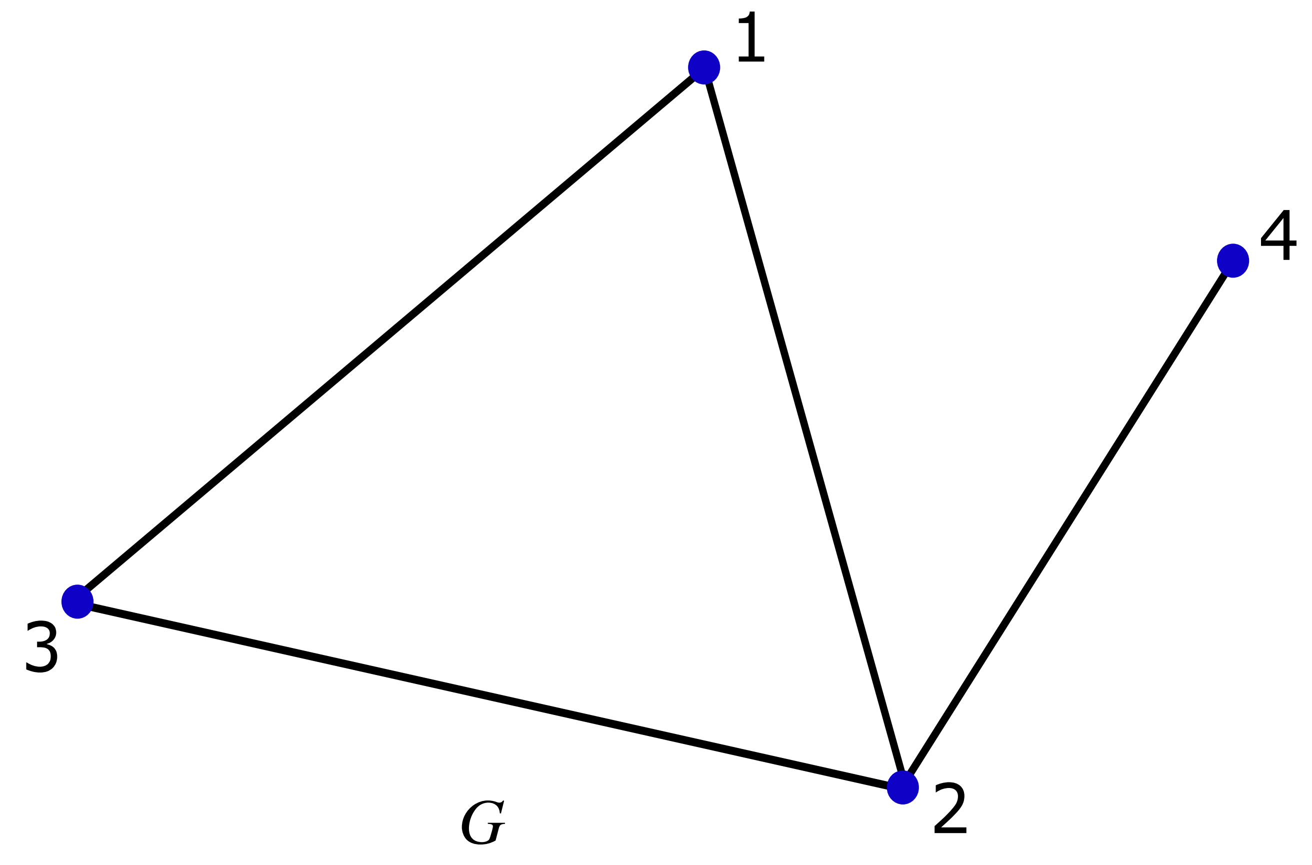
- Consensus dynamics for Euclidean space:
  - $x_i(0) = z_i$
  - $x_i^+ = x_i + \epsilon \sum_{j \sim i} (x_j - x_i)$
- Simplest consensus dynamics for a manifold [Tron et al, 2013]:
  - $x_i(0) = z_i$
  - $x_i^+ = \exp_{x_i}\left(\epsilon \sum_{j \sim i} \log_{x_i}(x_j)\right)$

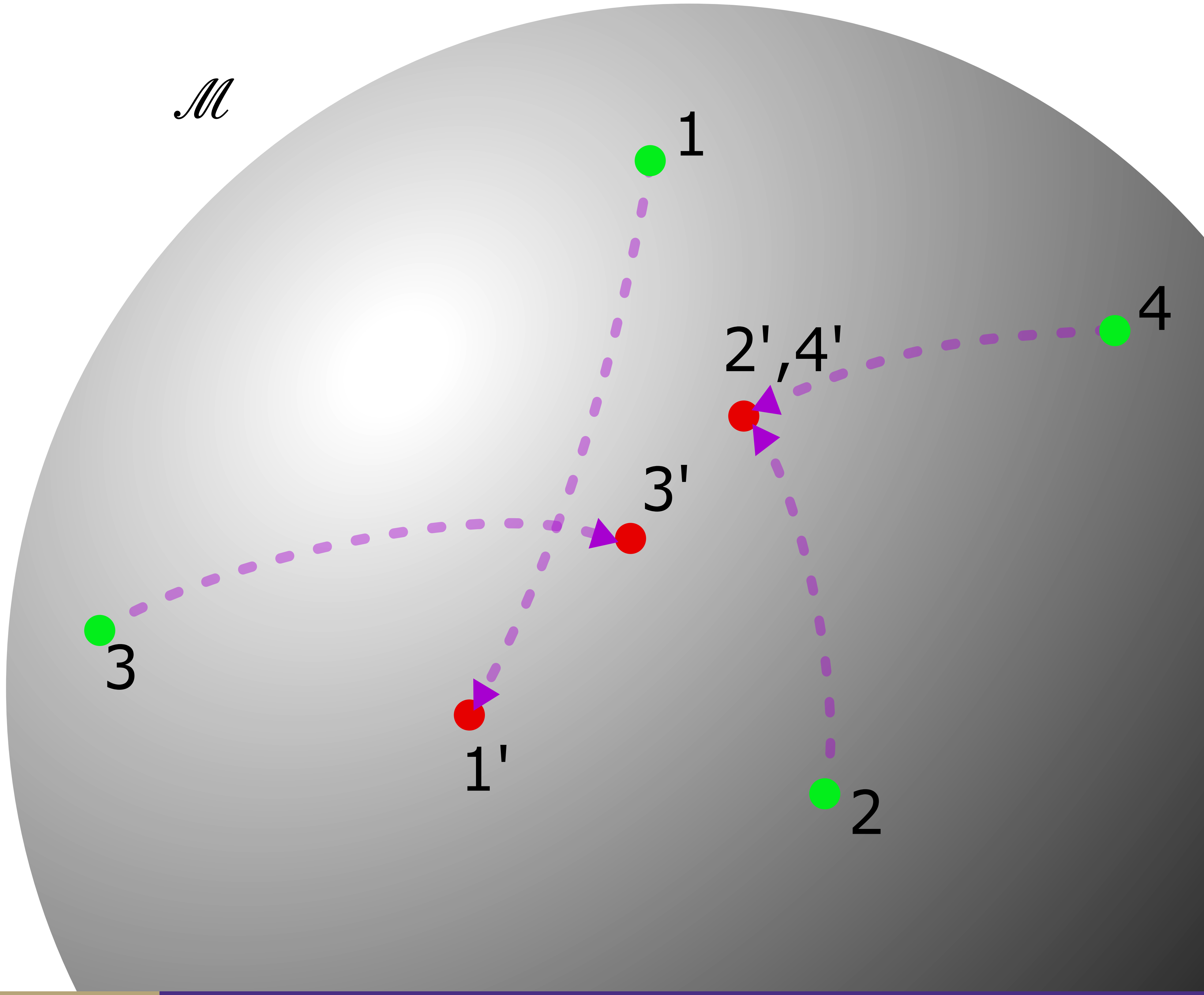
theorem: if $\{x_i(0)\}$ are initialized within a geodesically convex ball B and $\{x_i(k)\} \subset B$ for all k, then consensus is guaranteed.



$$x_i(k+1) = \begin{cases} \mathrm{RCM}(x_j(k) : j \sim i) & i - 1 \equiv k \pmod{N} \\ x_i(k) & \text{else} \end{cases}$$

- **intuition:** each agent moves towards neighbors' average
- **Round robin approach**: one agent moves per time step





guarantees

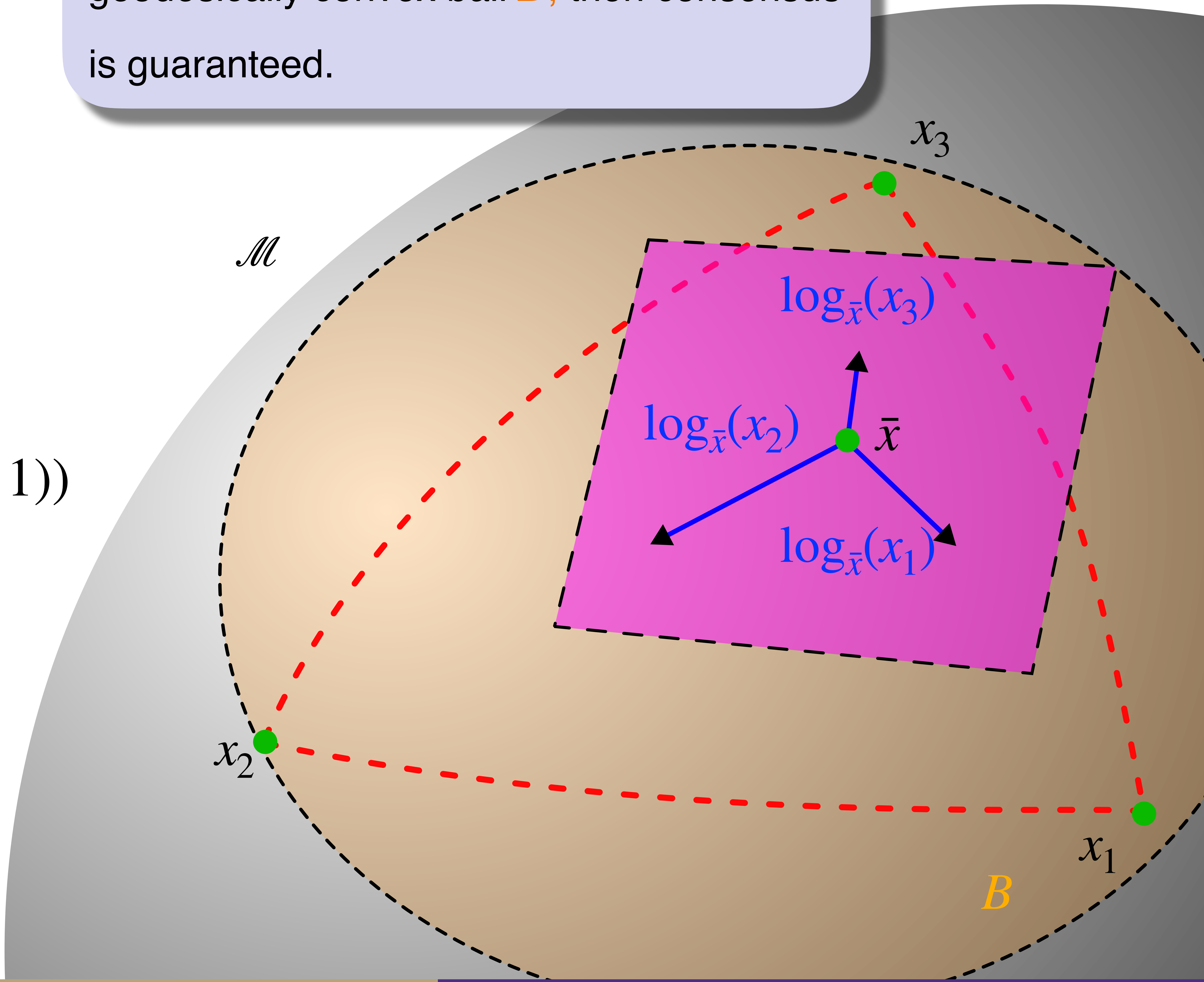
- $x_i(k+1)$ is in strict interior of geodesic convex hull of $\{x_j(k) : j \sim i\}$ (B) **(not true for extrinsic mean)**

- geodesic variance

$$\varphi(\mathbf{x}(k)) = \frac{1}{2} \sum_{\{i,j\} \in E} d_g(x_i(k), x_j(k))^2 > \varphi(\mathbf{x}(k+1))$$

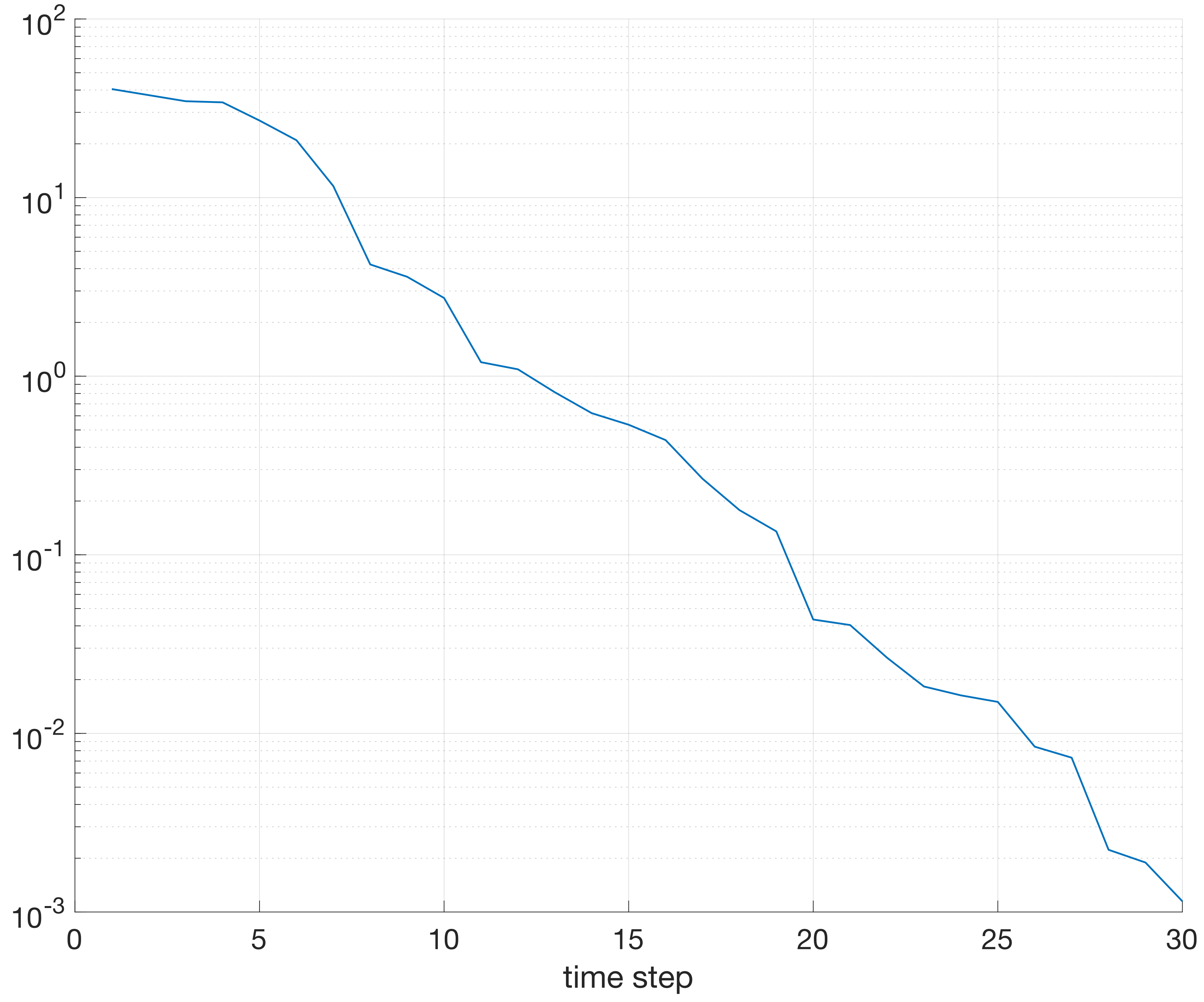
- $\varphi(\mathbf{x}(k+N)) = \varphi(\mathbf{x}(k))$ iff $x_1(k) = \ldots = x_N(k)$

theorem: if $\{x_i(0)\}$ are initialized within a geodesically convex ball B, then consensus is guaranteed.

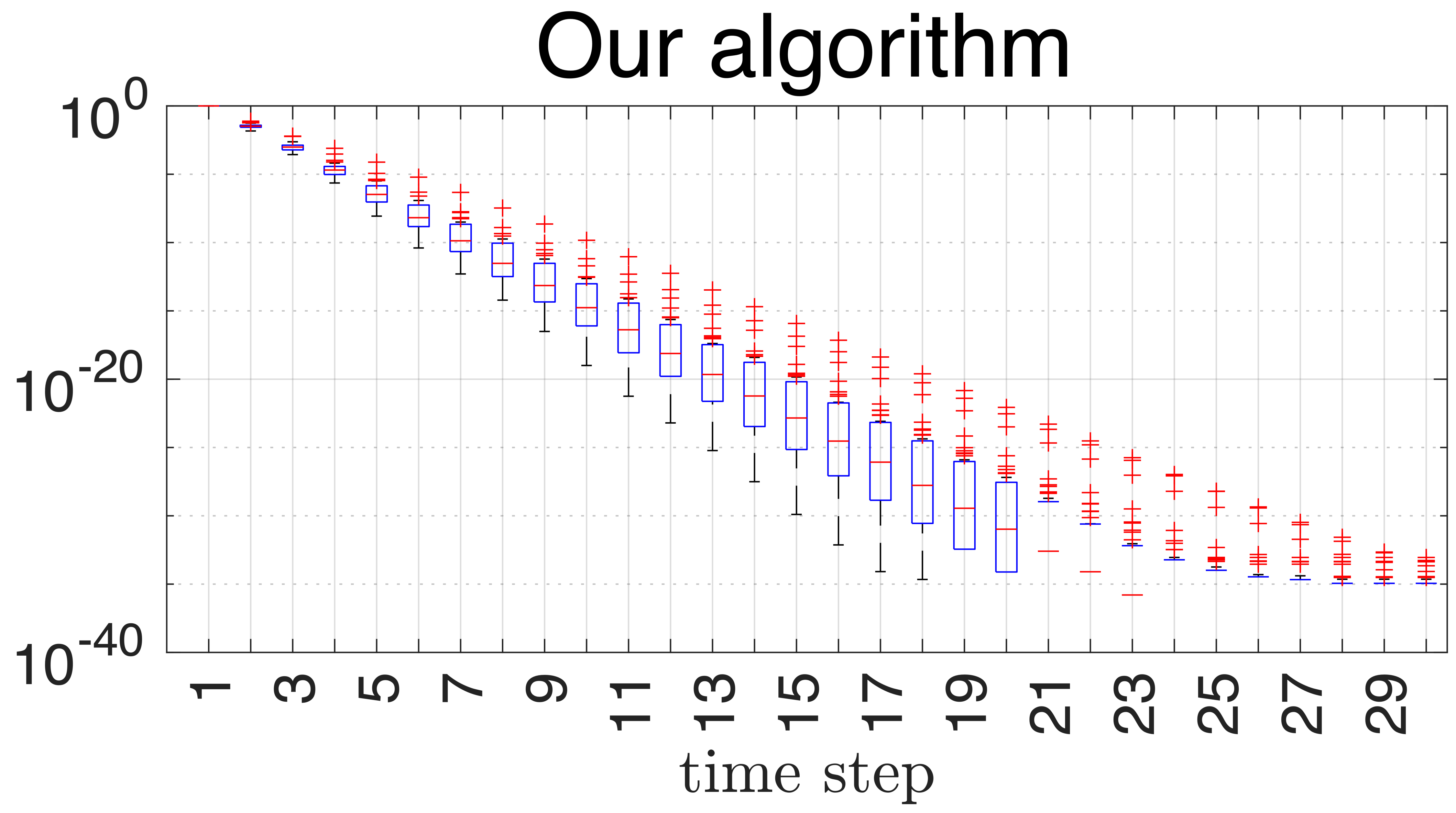


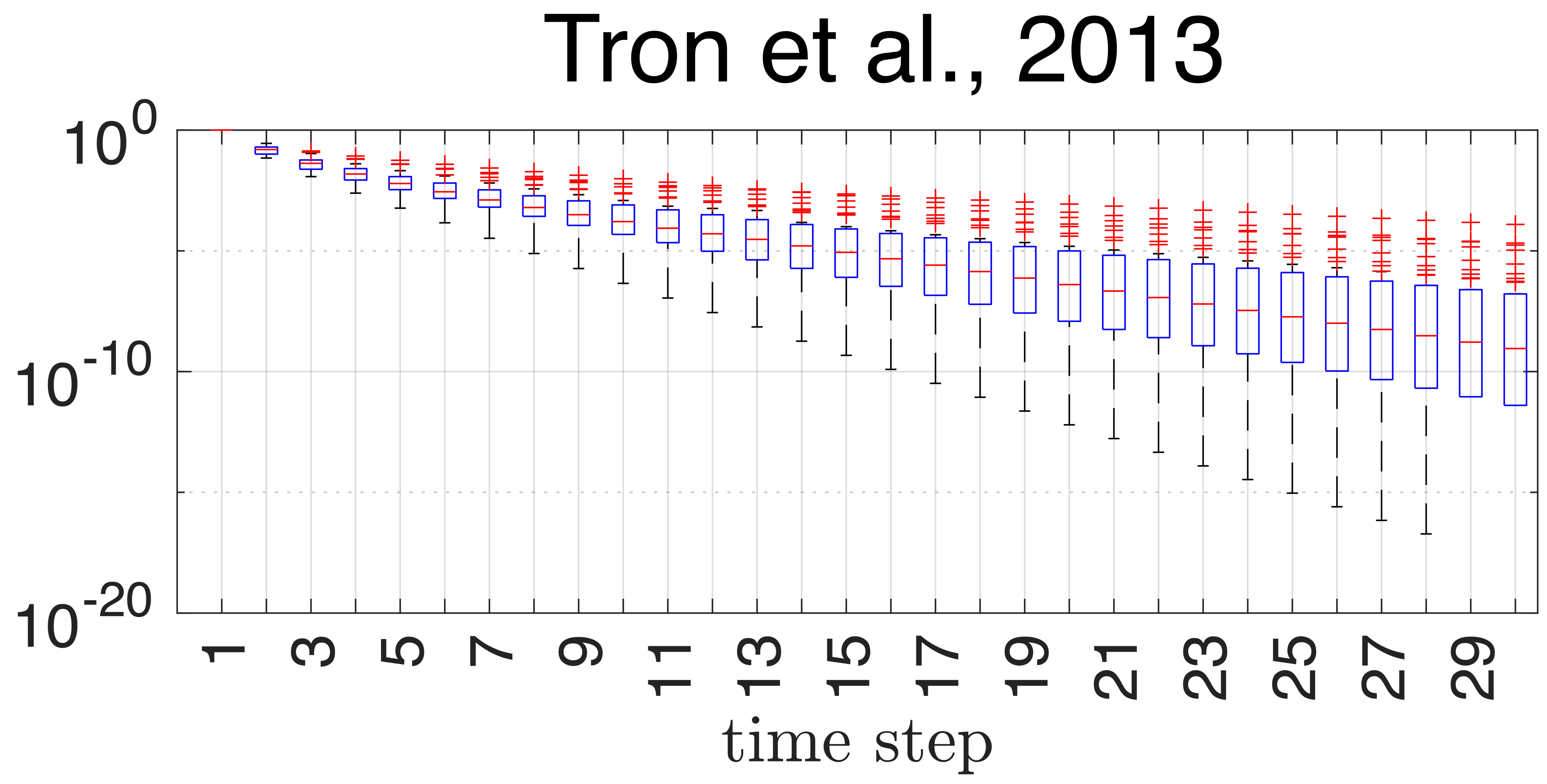
numerical simulation

- average cost over 40 runs
  - our algorithm (top) vs [2] (bottom)



linear decay of geodesic variance





concluding remarks

Euclidean consensus

dynamics:

$$x_i^+ = x_i + \epsilon \sum_{j \sim i} (x_j - x_i)$$

Results

$$\lim_{k \to \infty} x_i(k) = \frac{1}{N} \sum_{i=1}^{N} z_i$$

Riemannian consensus

dynamics:

$$x_i^+ = \exp_{x_i}\left(\epsilon \sum_{j \sim i} \log_{x_i}(x_j) \right)$$

Results

$$\lim_{k \to \infty} x_i(k) = ? \neq \text{RCM}(z_1, \dots, z_N)$$

Goal: Given $z_1, \dots, z_N \in M$ and a communication graph $\mathscr{G} = ([N], E)$ find discrete dynamics

$$x_i(k+1) = F_i\left(x_j(k) : j \in N_i \cup \{i\} \right)$$

such that

$$\lim_{k \to \infty} x_i(k) = RCM(z_1, \dots, z_N).$$

concluding remarks

- **in this talk:**
  - consensus on Riemannian manifolds
  - asynchronous approach with convergence guarantees
- **to be continued:**
  - agents move in random order
  - consensus to the Riemannian center of mass (CDC)