Trajectory Optimization on Smooth Manifolds
Intrinsic Successive Convexification

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$$\begin{cases} \min_{\mathbf{x},\mathbf{u}} & C(\mathbf{x},\mathbf{u}) := \sum_{k=0}^{N-1} \phi(x_k, u_k) + \phi_f(x_N) \\ \text{s.t.} & x_{k+1} = f(x_k, u_k), \\ & g(x_k, u_k) \le 0, \; x_0 \text{ given} \end{cases}$$

What if we know the state space is a *smooth manifold* $\mathcal{M} \subset \mathbb{R}^N$ with $\dim(\mathcal{M}) < N$? Can we take advantage of geometry and lower-dimensional state space?

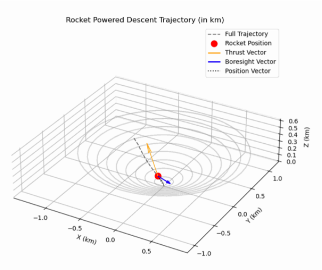


Figure: Powered descent $\mathbb{R}^3 \times SE(3) \times \mathbb{R}^3 \times \mathbb{R}^3$

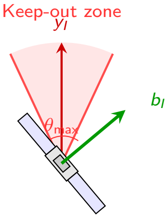


Figure: Satillite attitude $SO(3)$

Outline of Successive Convexification (SCvx)

- First-order solver for constrained optimal control problems
- Allows infeasible initial trajectories
- Highly scalable

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{u}} & C(\mathbf{x},\mathbf{u}) \\ \text{s.t.} & x_{k+1} = f(x_k, u_k) \\ & g(x_k, u_k) \leq 0 \end{cases}$$

$\downarrow$ convexify about $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$

$$\begin{cases} \min\limits_{\delta\mathbf{x},\,\delta\mathbf{u},\,\mathbf{v},\,\mathbf{s}} & C(\bar{\mathbf{x}} + \delta\mathbf{x},\ \bar{\mathbf{u}} + \delta\mathbf{u}) + \lambda P(\mathbf{v}, \mathbf{s}) \\ \text{s.t.} & f(x_k, u_k) - x_{k+1} + A_k \delta x_k + B_k \delta u_k \leq s_k \\ & g(x_k, u_k) + S_k \delta x_k + T_k \delta u_k = v_k \\ & s \geq 0, \quad \|\delta x\| \leq r \end{cases}$$

$\downarrow$

$$(\delta\mathbf{x}^*, \delta\mathbf{u}^*)$$

From Shortcomings to Goals

Shortcomings:

1. Different parameterizations give different solutions
2. Redundant dimensions
   - Ex. $\dot{q} = \frac{1}{2} q \otimes \omega$, $A_k \in \mathbb{R}^{4\times 4}$ instead of $\in \mathbb{R}^{3\times 3}$
3. Next trajectory won't be on manifold: $\bar{q} \leftarrow \bar{q} + \delta\bar{q}$

Goals:

1. Generalize SCvx to be *intrinsic* to the manifold, not the parameterization
2. Intelligent perturbations: ensure perturbations are tangent to the manifold surface
3. Use a retraction to "add" the perturbations to the trajectory in an intrinsic way

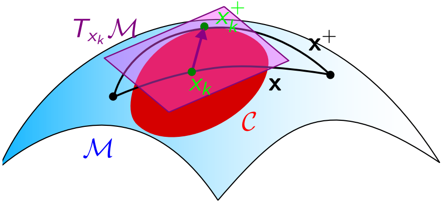


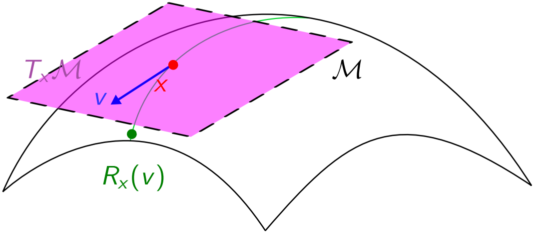
Figure: Visualization of perturbing a trajectory along a tangent space.

Riemannian Manifold

Riemannian manifold $\mathcal{M}$

- A space of points equipped with a cover of local coordinate systems
- Geodesic distance between points
- Tangent space $T_x\mathcal{M} \cong \mathbb{R}^n$
    - $T_q\mathcal{Q} = \{v \in \mathbb{R}^4 : q^T v = 0\} \cong \mathbb{R}^3$
- Calculus can be done on manifolds with necessary changes

Retraction: $R_x : T_x\mathcal{M} \to \mathcal{M}$



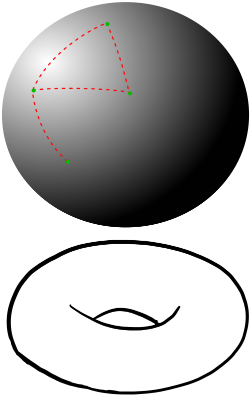


Figure: Smooth manifold examples

SCvx vs. Intrinsic SCvx: Key Differences

SCvx ($x \in \mathbb{R}^N, u \in \mathbb{R}^U$)	**iSCvx** ($x \in \mathcal{M}^n, u \in \mathcal{U}^m$)
Uses Jacobians $$A := D_x f(x,u) \in \mathbb{R}^{N\times N},$$ $$B := D_u f(x,u) \in \mathbb{R}^{N\times M}$$ in the ambient space ($N \geq n,\ M \geq m$).	Uses intrinsic differentials $$d_x f_{(x,u)} : T_x\mathcal{M} \to T_{f(x,u)}\mathcal{M},$$ $$d_u f_{(x,u)} : T_u\mathcal{U} \to T_{f(x,u)}\mathcal{M},$$ $$\mathbf{A} := [D_x f(x,u)\|_{T_x\mathcal{M}}] \in \mathbb{R}^{n\times n}$$ $$\mathbf{B} := [D_u f(x,u)\|_{T_u\mathcal{U}}] \in \mathbb{R}^{n\times m}$$

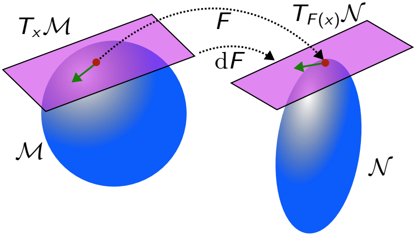


Figure: The differential $\mathrm{d}F_x : T_x\mathcal{M} \to T_{F(x)}\mathcal{N}$ of the function $F : \mathcal{M} \to \mathcal{N}$ at a point $x \in \mathcal{M}$

SCvx vs. Intrinsic SCvx: Key Differences (Cont.)

SCvx ($x \in \mathbb{R}^N, u \in \mathbb{R}^M$)	**iSCvx** ($x \in \mathcal{M}^n, u \in \mathcal{U}^m$)
Representation-dependent	Representation-invariant
Linearizes dynamics as $$\delta x_{k+1} \approx f(x_k, u_k) - x_{k+1} + A_k\,\delta x_k + B_k\,\delta u_k$$ (LTV, redundant dimensions, does not account for projection)	Intrinsic linearization: $$\delta x_{k+1} \approx R^{-1}_{x_{k+1}}(f(x_k, u_k)) + \mathbf{D}_k \circ (\mathbf{A}_k[\delta x_k] + \mathbf{B}_k[\delta u_k])$$ (LTV, minimal dimensions, accounts for projection)
Perturbations live in ambient $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_u}$ (redundant)	Perturbations live in $T_x\mathcal{M} \cong \mathbb{R}^n$ and $T_u\mathcal{U} \cong \mathbb{R}^m$ (minimal dimension)

SCvx vs. Intrinsic SCvx: Key Differences (Cont.)

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Perturbations live in ambient $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_u}$ (redundant)	Perturbations live in $T_x\mathcal{M} \cong \mathbb{R}^n$ and $T_u\mathcal{U} \cong \mathbb{R}^m$ (minimal dimension)
Local problem exploits cost convexity: $C(\mathbf{x} + \delta\mathbf{x}, \mathbf{u} + \delta\mathbf{u})$	Local problem exploits cost *geodesic* convexity: $\hat{C}^{(2)}_{(\mathbf{x},\mathbf{u})}(\delta\mathbf{x}, \delta\mathbf{u}) := C(\mathbf{x}, \mathbf{u}) + \mathrm{d}C_{(\mathbf{x},\mathbf{u})}[\delta\mathbf{x}, \delta\mathbf{u}] + \frac{1}{2}\nabla^2 C_{(\mathbf{x},\mathbf{u})}[\delta\mathbf{x}, \delta\mathbf{u}]$

SCvx vs. Intrinsic SCvx: Key Differences (Cont.)

SCvx ($x \in \mathbb{R}^N, u \in \mathbb{R}^M$)	**iSCvx** ($x \in \mathcal{M}^n, u \in \mathcal{U}^m$)
Representation-dependent	Representation-invariant
Linearizes dynamics as $$\delta x_{k+1} \approx f(x_k, u_k) - x_{k+1} + A_k\, \delta x_k + B_k\, \delta u_k$$ (LTV, redundant dimensions, does not account for projection)	Intrinsic linearization: $$\delta x_{k+1} \approx R^{-1}_{x_{k+1}}(f(x_k, u_k)) + \mathbf{D}_k \circ (\mathbf{A}_k[\delta x_k] + \mathbf{B}_k[\delta u_k])$$ (LTV, minimal dimensions, accounts for projection)
Perturbations live in ambient $\mathbb{R}^{n_x}$ and $\mathbb{R}^{n_u}$ (redundant)	Perturbations live in $T_x\mathcal{M} \cong \mathbb{R}^n$ and $T_u\mathcal{U} \cong \mathbb{R}^m$ (minimal dimension)
Local problem exploits cost convexity: $C(\mathbf{x} + \delta\mathbf{x}, \mathbf{u} + \delta\mathbf{u})$	Local problem exploits cost *geodesic* convexity: $\hat{C}^{(2)}_{(\mathbf{x},\mathbf{u})}(\delta\mathbf{x}, \delta\mathbf{u}) := C(\mathbf{x}, \mathbf{u}) + \mathrm{d}C_{(\mathbf{x},\mathbf{u})}[\delta\mathbf{x}, \delta\mathbf{u}] + \frac{1}{2}\nabla^2 C_{(\mathbf{x},\mathbf{u})}[\delta\mathbf{x}, \delta\mathbf{u}]$
State update: $\mathbf{x} \leftarrow \mathbf{x} + \delta\mathbf{x}$, $\mathbf{u} \leftarrow \mathbf{u} + \delta\mathbf{u}$ (not manifold-preserving)	State update: $\mathbf{x} \leftarrow R_{\mathbf{x}}(\delta\mathbf{x})$, $\mathbf{u} \leftarrow R_{\mathbf{u}}(\delta\mathbf{u})$ (manifold-preserving)

Outline of Intrinsic SCvx

$$\begin{cases} \min\limits_{\mathbf{x}\subset\mathcal{M},\mathbf{u}\subset\mathcal{U}} & C(\mathbf{x},\mathbf{u}) \\ \text{s.t.} & x_{k+1} = f(x_k, u_k) \\ & g(x_k, u_k) \leq 0 \end{cases}$$

$\downarrow$ convexify about $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$

$$\begin{cases} \min\limits_{\delta\mathbf{x},\,\delta\mathbf{u},\,\mathbf{v},\,\mathbf{s}} & \hat{C}^{(2)}_{(\bar{\mathbf{x}},\bar{\mathbf{u}})}(\delta\mathbf{x}, \delta\mathbf{u}) + \lambda P(\mathbf{v}, \mathbf{s}) \\ \text{s.t.} & R^{-1}_{x_{k+1}}(f(x_k, u_k)) + \mathbf{D}_k \circ (\mathbf{A}_k[\delta x_k] + \mathbf{B}_k[\delta u_k]) \leq s_k \\ & g(x_k, u_k) + \mathbf{S}_k[\delta x_k] + \mathbf{T}_k[\delta u_k] = v_k \\ & s \geq 0, \quad \|\delta x\| \leq r \end{cases}$$

$\downarrow$

$$(\delta\mathbf{x}^*, \delta\mathbf{u}^*)$$

Advantages

1. The convex sub-problem has dimension $\dim \mathcal{M}$, independent of the ambient representation.
    - E.g., quaternions (4 vars) or rotation matrices (9 vars) both yield an $n = 3$ LTV sub-problem.
2. The formulation is representation-invariant: any coordinate choice for $\mathcal{M}$ gives the same convex sub-problem.
3. Each updated trajectory remains on the manifold.
4. Projection onto $\mathcal{M}$ is handled intrinsically within the convexification step.
5. Convex sub-problem is feasible, as with SCvx.

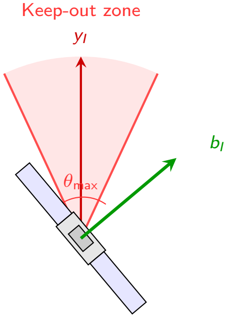
Example Problem: Constrained Attitude Control

Dynamics:

$$q_{k+1} = f(q_k, \omega_k) := q_k \otimes \exp(\Delta t \cdot \omega_k)$$

Retraction: $R_q(v) := q \otimes \exp\left(q^{-1} \otimes v\right)$
Input: Angular velocity $\omega_k \in \mathbb{R}^3$
State: Attitude $q_k \in \mathcal{Q}$
Constraints:

1. Keep-out zone:
   $g(q) := b_I \cdot y_I - \cos\theta_{\max}$,
   - y_I = inertial-fixed keep-out direction
   - b_B = body-fixed boresight direction
   - $\theta_{\max}$ = keep-out angular radius
2. Initial attitude: $q_0 \in \mathcal{Q}$
3. Target attitude: $q_{des} \in \mathcal{Q}$



- SCvx stage cost:

$$\phi(q_k, \omega_k) = \frac{1}{2}\|q_k - q_{des}\|_2^2 + \frac{1}{2}\|\omega_k\|_2^2$$

- iSCvx stage cost:

$$\phi(q_k, \omega_k) = \frac{1}{2} d_{\mathcal{Q}}(q_k, q_{des})^2 + \frac{1}{2}\|\omega_k\|_2^2$$

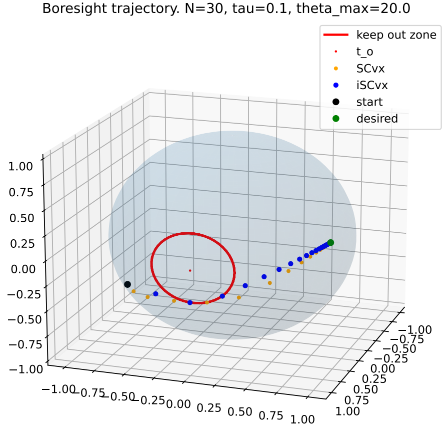


Figure: Trajectory of boresight vector, red circle is keep-out zone

TABLE I
SCVX AND ISCVX COMPARISON: CASE 1

	$\theta_{\max} = 10°$		$\theta_{\max} = 30°$	
	SCvx	**iSCvx**	**SCvx**	**iSCvx**
Avg. Iter.	40.21	24.89	45.8	26.8
Std. Iter.	9.23	2.14	18.28	1.88
Time (s)	6.26	4.40	7.10	4.70
Avg. Geo. Cost	4.98	4.73	6.50	5.67
Avg. Eucl. Cost	7.21	6.91	9.65	8.17

Results compare SCvx and iSCvx for spacecraft attitude control under different angle constraints with $N = 30$ time steps and discretization rate $\Delta t = 0.1$ sec. Values are averages over multiple runs.

TABLE II
SCVX AND ISCVX COMPARISON: CASE 2

	$\theta_{\max} = 10°$		$\theta_{\max} = 30°$	
	SCvx	**iSCvx**	**SCvx**	**iSCvx**
Avg. Iter.	67.9	24.75	65.72	25.65
Std. Iter.	34.86	2.22	17.02	2.45
Time (s)	22.18	9.09	21.43	9.37
Avg. Geo. Cost	9.04	8.96	10.59	10.50
Avg. Eucl. Cost	11.94	13.34	13.98	15.42

$N = 60$ time steps and discretization rate $\Delta t = 0.05$ sec.

Summary and Future Work

Summary: Intrinsic SCvx is an SCvx-like algorithm formulated directly on the state-space manifold, eliminating redundant ambient dimensions and yielding a representation-invariant optimal control solving methodology

Future work:

- Extend existing convergence guarantees for SCvx to the iSCvx
- Develop an in-depth tutorial of iSCvx with simple worked examples to make the method more accessible to practitioners
- Extend and refine the open-source iSCvx library